The transition point in the zero noise limit for a 1D Peano example

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Dedicated to the memory of José Real

Abstract

The zero-noise result for Peano phenomena of Bafico and Baldi (1982) is revisited. The original proof was based on explicit solutions to the elliptic equations for probabilities of exit times. The new proof given here is purely dynamical, based on a direct analysis of the SDE and the relative importance of noise and drift terms. The transition point between noisy behavior and escaping behavior due to the drift is identified.

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1 Introduction

Given a probability space $(\Omega, \mathcal{A}, P)$, consider the SDE

$$dX_t^{x,\varepsilon} = b\left(X_t^{x,\varepsilon}\right) dt + \varepsilon dW_t, \qquad X_0^{x,\varepsilon} = x \tag{1}$$

in $\mathbb{R}^d$ with $b \in C^\alpha\left(\mathbb{R}^d, \mathbb{R}^d\right)$ and at most linear growth at infinity, $(W_t)_{t\geq 0}$ being a Brownian motion in $\mathbb{R}^d$. When $\varepsilon > 0$, a strong solution exists and it is pathwise unique, hence also unique in law (see [11], [9], [6], [5]); we denote its law by P_x^ε. When $\varepsilon = 0$, namely for the deterministic Cauchy problem

$$\frac{dX_t^x}{dt} = b\left(X_t^x\right), \qquad X_0^x = x, \tag{2}$$

a solution exists by Peano theorem but uniqueness may fail, like in the example studied below. In absence of uniqueness, a relevant question is selection: does the limit of P_x^ε as $\varepsilon \to 0$ concentrate on special solutions, which, because of this property, may be considered more "physical" than others?

Under the previous assumptions on b, the family $\{P_x^\varepsilon\}_{\varepsilon\in(0,1)}$ is tight and each limit measure μ is supported by the non empty closed set $\mathcal{C}$ of continuous solutions $(X_t^x)_{t\geq 0}$ to the ODE (2). As we just said, the set $\mathcal{C}$ is then expected to contain more than one element. The question is then to determine the limit measures μ of the family $\{P_x^\varepsilon\}_{\varepsilon\in(0,1)}$ together with the elements of $\mathcal{C}$ which are *selected* by these limit measures.

Precise general results on this problem, in $\mathbb{R}^d$, are not known, except for structural facts as in [2], [3] (however, see [4] for a special 2D example). The main and striking paper is [1], which gives a very general and explicit solution in the case $d = 1$, b having one singular point x_0 ($b(x_0) = 0$, b being not locally Lipschitz around x_0). The proof makes essential use of explicit solutions to elliptic equations for the exit time from an interval and for ruin type probabilities. This powerful analytic approach allows [1] to treat a large variety of examples. However, it is also the main restriction in the attempt to extend the theory to higher dimensions. Indeed, the elliptic equations in $d = 1$ are second order linear ordinary differential equations, hence explicitly solvable. In higher dimensions this approach fails, unless special symmetries allow to reduce the dimension. Concerning one of the main examples of [1], namely (3) studied below, we also refer to the very interesting works [7] and [8], where large deviations are investigated and atypical results are discovered.

It would be thus desirable to have other approaches to the same problem. This motivated us to develop the new proof given here. Let us however insist that these remarks are only a general motivation for this research, since we do not solve the problem of a generalization to higher dimensions; we only give a new proof based on dynamical considerations.

We give here a new proof of the result of Bafico and Baldi [1], in the particular case (always $d = 1$) of the drift

$$b(x) = \begin{cases} A^+ |x|^\alpha & \text{for} \quad x \geq 0 \\ -A^- |x|^\alpha & \text{for} \quad x < 0 \end{cases} \tag{3}$$

for some $\alpha \in (0, 1)$ and $A^+, A^- > 0$. The result is that P_0^ε, the law of the solution of the SDE with initial condition $x = 0$, weakly converges to a combination of delta Dirac masses at the extreme solutions of the ODE:

Theorem 1

$$P_0^\varepsilon \to p^+ \delta_{x_\cdot^+} + p^- \delta_{x_\cdot^-},$$

in the weak sense, as $\varepsilon \to 0$, *where*

$$p^+ = \frac{(A^-)^{-\frac{1}{1+\alpha}}}{(A^+)^{-\frac{1}{1+\alpha}} + (A^-)^{-\frac{1}{1+\alpha}}}, \quad p^- = \frac{(A^+)^{-\frac{1}{1+\alpha}}}{(A^+)^{-\frac{1}{1+\alpha}} + (A^-)^{-\frac{1}{1+\alpha}}},$$

and

$$x_t^+ = C_{\alpha,A^+} t^{\frac{1}{1-\alpha}}, \quad x_t^- = -C_{\alpha,A^-} t^{\frac{1}{1-\alpha}},$$

with $C_{\alpha,A^\pm} = \left(A^\pm (1-\alpha)\right)^{1/(1-\alpha)}$.

The point of the new proof given here is its *dynamical character*. The identification of the paths which are selected is not based on some auxiliary PDEs as in [1] but only on the dynamical properties of the SDE (1). Only the computation of the precise weights in the combination of the Dirac masses involves a PDE through a martingale argument. The new proof thus gives some new insight into the actual behavior of solutions, which is not visible in the PDE approach. Precisely, we identify the existence of two regimes. At the beginning of time, the solution which started from $x = 0$ behaves like the Brownian motion $(\varepsilon W_t)_{t \geq 0}$, although ε is very small, because the drift is much smaller (this happens also for a Lipschitz drift). But close to the time-space points

$$(t_\varepsilon, x_\varepsilon) := \left(\varepsilon^{\frac{2(1-\alpha)}{1+\alpha}}, x_{t_\varepsilon}^\pm\right) = \left(\varepsilon^{\frac{2(1-\alpha)}{1+\alpha}}, \pm C_{\alpha,A^\pm} \varepsilon^{\frac{2}{1+\alpha}}\right) \tag{4}$$

a transition occurs: the drift becomes much stronger than the noise (the usual fluctuations of the noise do not contrast the drift anymore) and pushes the trajectories far away from the neighborhood of $x = 0$, roughly along one of the trajectories $(x_t^\pm)_{t \geq 0}$.

The easiest way to identify heuristically the transition time-space point (4) is to compare x_t^+ (or x_t^-) with εW_t. Forgetting the scale constants $C_{\alpha,A^\pm}$ in the definition of x_t^+ and x_t^-, we thus claim that the typical time t at which transition is given by the solution of the equation

$$t^{\frac{1}{1-\alpha}} = \varepsilon t^{\frac{1}{2}}. \tag{5}$$

We then get $t_\varepsilon := \varepsilon^{\frac{2(1-\alpha)}{1+\alpha}}$ as typical transition time. Accordingly, we then define

$$x_\varepsilon := t_\varepsilon^{\frac{1}{1-\alpha}} = \varepsilon^{\frac{2}{1+\alpha}}$$

as the typical space scale for observing the transition between the two regimes. This simple intuition is confirmed by the proofs below.

We emphasize that the same method would permit to handle completely asymmetric cases for which $b(x)$ has the form $b(x) = A^+|x|^{\alpha^+}$ for $x \geq 0$ and $b(x) = -A^-|x|^{\alpha^-}$ for $x < 0$ with $\alpha^+ \neq \alpha^-$. In such cases, the two extremal paths do not generate the same transition times so that only one of them is selected, namely the one driven by the smallest exponent. Intuitively, this amounts to letting the ratio A^+/A^- tend to 0 or $+\infty$ in our case and thus letting the weights in the Dirac combination tend to $(0,1)$ or $(1,0)$.

We feel appropriate to state, right here in the introduction, our two main results concerning the transition point, because they are our main contribution to the understanding of this dynamical problem. For every $r > 0$, let us denote by τ_r the exit time from $(-r, r)$, defined on the canonical space $C\left([0,+\infty);\mathbb{R}\right)$, as

$$\tau_r\left(\xi\right) = \inf\left\{t > 0 : |\xi_t| \geq r\right\}, \qquad \xi \in C\left([0,+\infty);\mathbb{R}\right),$$

when this set is not empty, $\tau_r\left(\xi\right) = +\infty$ otherwise. We denote by P_x^ε the law of $X^{x,\varepsilon}$ and by E_x^ε the corresponding expectation.

Proposition 2 *For every function $\widetilde{t}_\varepsilon > t_\varepsilon$ such that $\lim_{\varepsilon\to 0} \widetilde{t}_\varepsilon / t_\varepsilon = +\infty$, we have*

$$\lim_{\varepsilon\to 0} \sup_{x\in(-x_\varepsilon, x_\varepsilon)} P_x^\varepsilon\left(\tau_{x_\varepsilon} > \widetilde{t}_\varepsilon\right) = 0.$$

The proposition states that with high probability the system reaches $\pm x_\varepsilon$ in a time just a little greater than t_ε, when it starts inside $(-x_\varepsilon, x_\varepsilon)$. This fact is mainly due to the fluctuations of the noise, the drift playing a negligible role on a time interval of length of the same order as t_ε. Put it differently, until time t_ε, the noise dominates. We prove this result in any dimension, under quite general conditions.

Then we show that, starting from x_ε or above, the solution remains above x_t^+ forever, with probability larger than some $\lambda > 0$ (similarly if it starts from $-x_\varepsilon$). More precisely, it remains above $(1-\gamma)\, x_t^+$ for any arbitrarily prescribed $\gamma \in (0,1)$. The result is complemented by the following fact: for every function $\widetilde{x}_\varepsilon > x_\varepsilon$ such that $\lim_{\varepsilon\to 0} \widetilde{x}_\varepsilon / x_\varepsilon = +\infty$, if we start above $\widetilde{x}_\varepsilon$ then the solution remains above x_t^+ forever, with probability close to one. The formal statement is:

Theorem 3 *Let $\gamma \in (0,1)$ be given. Then there exists a constant $\lambda_\gamma > 0$, independent of $\varepsilon \in (0,1)$, such that*

$$\inf_{x \geq x_\varepsilon} P\left(X_t^{x,\varepsilon} \geq (1-\gamma)\, x_t^+, \quad \forall t \geq 0\right) \geq \lambda_\gamma > 0.$$

Moreover, for every function $\widetilde{x}_\varepsilon > x_\varepsilon$ such that $\lim_{\varepsilon\to 0} \widetilde{x}_\varepsilon / x_\varepsilon = +\infty$, we have

$$\lim_{\varepsilon\to 0} \inf_{x \geq \widetilde{x}_\varepsilon} P\left(X_t^{x,\varepsilon} \geq (1-\gamma)\, x_t^+, \quad \forall t \geq 0\right) = 1.$$

Similar results hold for $(x_t^-)_{t\geq 0}$.

This fact will imply our final result. We emphasize that it occurs because of the drift. In particular, a Lipschitz continuous drift would not give such a result: Observe for instance that the transition time, as defined by (5), tends to 1 as α tends to 1, saying that no transition occurs in small time when $\alpha = 1$.

These two steps lead to the solution of our problem. Indeed, with large probability, we reach $\pm x_\varepsilon$ in a very short time of order $\widetilde{t}_\varepsilon$ (Proposition 2). Then, iterating the argument of escape with probability larger than λ_γ, we reach a prescribed $\pm\widetilde{x}_\varepsilon$ with probability close to one, again in a short time (Corollary 8). Finally, restarting from $\pm\widetilde{x}_\varepsilon$, we escape above or below $(1-\gamma)\, x_t^\pm$ forever (Theorem 3). This shows that only the extremal paths are selected at the limit. The weights in the combination of the delta Dirac masses appearing in the statement of Theorem 1 are then computed by a martingale argument.

Our hope is that we have identified new ideas behind the zero-noise limit problem which may be extended to other examples and higher dimension, but for the time being we need to restrict ourselves to the special 1D case above.

2 Proof of Proposition 2

We give two proofs. The first one, which is purely dynamical, is true in any dimension under moderate assumptions. This may be an indication that the dynamical proof presented here is promising for generalizations. The second one is a refinement of the first one; it relies on the exact scaling properties of the SDE at hand.

2.1 Multidimensional result

Let $b : \mathbb{R}^d \to \mathbb{R}^d$ be a function such that

$$|b(x)| \le M r^{\alpha} \qquad \forall x \in \mathbb{R}^d, \forall r > 0, \text{ with } |x| \le r,$$

for some $M > 0$, $\alpha \in (0,1)$. For a given $x \in \mathbb{R}^d$ and every $\varepsilon \in (0,1)$, let $(X_t^{x,\varepsilon})_{t\ge 0}$ be the solution of

$$dX_t^{x,\varepsilon} = b\left(X_t^{x,\varepsilon}\right) dt + \varepsilon dW_t, \qquad X_0^{x,\varepsilon} = x$$

where $(W_t)_{t\ge 0}$ is a Brownian motion in $\mathbb{R}^d$ and let P_x^{ε} be its law on the space of continuous paths. As above, set

$$r_{\varepsilon} := \varepsilon^{\frac{2}{1+\alpha}}, \qquad t_{\varepsilon} := \varepsilon^{\frac{2(1-\alpha)}{1+\alpha}},$$

and

$$\tau_r(\xi) := \inf\{t > 0 : |\xi_t| \ge r\} \quad \text{for any } r > 0 \text{ and } \xi \in C([0,+\infty); \mathbb{R}^d),$$

with the usual convention for the infimum of an empty set.

Proposition 4 *One has*

$$\sup_{|x|\le r_{\varepsilon}} P_x^{\varepsilon}\left(\tau_{r_{\varepsilon}} > t_{\varepsilon}\right) \le \theta := P\left(|Z| \le 2 + M\right)$$

where $Z \sim N(0, Id)$.

Proof. If $\tau_{r_{\varepsilon}}(X^{x,\varepsilon}) > t_{\varepsilon}$ then $|X_t^{x,\varepsilon}| \le r_{\varepsilon}$ for $t \in [0, t_{\varepsilon}]$. Hence, from

$$X_t^{x,\varepsilon} = x + \int_0^t b\left(X_s^{x,\varepsilon}\right) ds + \varepsilon W_t,$$

with $|x| \le r_{\varepsilon}$, we get

$$\begin{aligned}\varepsilon |W_t| &\le |X_t^{x,\varepsilon}| + |x| + \int_0^t \left|b\left(X_s^{x,\varepsilon}\right)\right| ds \\ &\le 2r_{\varepsilon} + tMr_{\varepsilon}^{\alpha} \le 2r_{\varepsilon} + t_{\varepsilon} M r_{\varepsilon}^{\alpha}\end{aligned}$$

for $t \in [0, t_\varepsilon]$. Moreover,

$$t_\varepsilon r_\varepsilon^\alpha = r_\varepsilon.$$

Hence

$$\varepsilon \left| W_t \right| \leq (2 + M)\, r_\varepsilon \qquad \text{for } t \in [0, t_\varepsilon]\,.$$

In particular, this implies $\varepsilon \left| W_{t_\varepsilon} \right| \leq (2 + M)\, r_\varepsilon$. Since the random vector $Z := t_\varepsilon^{-1/2} W_{t_\varepsilon}$ is $N(0, Id)$, we deduce that

$$\begin{aligned} P\big(\varepsilon \left| W_{t_\varepsilon} \right| \leq (2 + M)\, r_\varepsilon\big) &= P\left(\left| t_\varepsilon^{-1/2} W_{t_\varepsilon} \right| \leq \frac{(2 + M)\, r_\varepsilon}{\varepsilon t_\varepsilon^{1/2}} \right) \\ &= P\left(|Z| \leq 2 + M \right) =: \theta < 1. \end{aligned}$$

We have used the identity

$$r_\varepsilon = \varepsilon t_\varepsilon^{1/2}.$$

We have proved

$$P_x^\varepsilon \left(\tau_{r_\varepsilon} > t_\varepsilon \right) \leq \theta.$$

The proof is complete. ■

Corollary 5 *Assume that, for every $\varepsilon \in (0, 1)$, the solutions $(X_t^{x,\varepsilon})_{t \geq 0}$ are a strong Markov family w.r.t. the initial condition. Then, for any integer $n \geq 1$,*

$$\sup_{|x| \leq r_\varepsilon} P_x^\varepsilon \left(\tau_{r_\varepsilon} > n t_\varepsilon \right) \leq \theta^n.$$

In particular, for every function $\widetilde{t}_\varepsilon > t_\varepsilon$ such that $\lim_{\varepsilon \to 0} \widetilde{t}_\varepsilon / t_\varepsilon = +\infty$, we have

$$\lim_{\varepsilon \to 0} \sup_{|x| \leq r_\varepsilon} P_x^\varepsilon \left(\tau_{r_\varepsilon} > \widetilde{t}_\varepsilon \right) = 0.$$

Proof. We know that

$$P_x^\varepsilon \left(\tau_{r_\varepsilon} > t_\varepsilon \right) \leq \theta = P\left(|Z| \leq 2 + M \right).$$

We consider the sequence of stopping times

$$\tau_{r_\varepsilon}^{(n)} := (n t_\varepsilon) \wedge \tau_{r_\varepsilon}$$

for $n = 0, 1, ...$ and apply strong Markov property to get

$$\begin{aligned} P_x^\varepsilon\left(\tau_{r_\varepsilon} > nt_\varepsilon\right) &= P_x^\varepsilon\left(\tau_{r_\varepsilon} > nt_\varepsilon \middle| |\xi_{\tau_{r_\varepsilon}^{(n-1)}}| < r_\varepsilon\right) P_x^\varepsilon\left(|\xi_{\tau_{r_\varepsilon}^{(n-1)}}| < r_\varepsilon\right) \\ &\quad + P_x^\varepsilon\left(\tau_{r_\varepsilon} > nt_\varepsilon \middle| |\xi_{\tau_{r_\varepsilon}^{(n-1)}}| = r_\varepsilon\right) P_x^\varepsilon\left(|\xi_{\tau_{r_\varepsilon}^{(n-1)}}| = r_\varepsilon\right) \\ &\le \theta P_x^\varepsilon\left(\tau_{r_\varepsilon} > (n-1)\, t_\varepsilon\right), \end{aligned}$$

because $P_x^\varepsilon\left(\tau_{r_\varepsilon} > nt_\varepsilon \middle| |\xi_{\tau_{r_\varepsilon}^{(n-1)}}| = r_\varepsilon\right) = 0$ and $P_x^\varepsilon\left(\tau_{r_\varepsilon} > nt_\varepsilon \middle| |\xi_{\tau_{r_\varepsilon}^{(n-1)}}| < r_\varepsilon\right) \le \theta$. Therefore

$$P_x^\varepsilon\left(\tau_{r_\varepsilon} > nt_\varepsilon\right) \le \theta^n.$$

The proof is complete. ■

2.2 Proof by exact scaling

We also provide this proof because it provides the rate of convergence to zero of $P_x^\varepsilon\left(\tau_{x_\varepsilon} > \widetilde{t}_\varepsilon\right)$. However, it is restricted to the 1D case, since it relies on the perfect scaling property of the drift b in (3).

Notice, by standard recurrence properties of the 1D Brownian motion, that $\|E_\cdot^1[\tau_{x_1}]\|_\infty < \infty$

Lemma 6 *Consider the 1D case with b given by (3) and recall that $x_\varepsilon = \varepsilon^{\frac{2}{1+\alpha}}$ and $t_\varepsilon = \varepsilon^{\frac{2(1-\alpha)}{1+\alpha}}$. Then, for all $x \in (-x_\varepsilon, x_\varepsilon)$, the law of τ_{x_ε} under P_x^ε is the same as the law of $t_\varepsilon \tau_{x_1}$ under $P^1_{x_\varepsilon^{-1}x}$. In particular,*

$$P_x^\varepsilon\big(\tau_{x_\varepsilon} > \tilde{t}_\varepsilon\big) = P^1_{x_\varepsilon^{-1}x}\Big(\tau_1 > \frac{\tilde{t}_\varepsilon}{t_\varepsilon}\Big) \le \frac{t_\varepsilon}{\widetilde{t}_\varepsilon} \left\|E_\cdot^1[\tau_{x_1}]\right\|_\infty.$$

Proof. We notice that

$$d\big(x_\varepsilon^{-1} X_{t_\varepsilon t}\big) = x_\varepsilon^{-1} t_\varepsilon b(X_{t_\varepsilon t})dt + x_\varepsilon^{-1} t_\varepsilon^{\frac{1}{2}} \varepsilon d\hat{W}_t^\varepsilon,$$

for another Brownian motion $(\hat{W}_t^\varepsilon)_{t\ge 0}$. It is well-checked that

$$x_\varepsilon^{-1} t_\varepsilon^{\frac{1}{2}} \varepsilon = 1.$$

Moreover, by scaling property of b,

$$x_\varepsilon^{-1} t_\varepsilon b(X_{t_\varepsilon t}) = x_\varepsilon^{\alpha-1} t_\varepsilon b\big(x_\varepsilon^{-1} X_{t_\varepsilon t}\big) = b\big(x_\varepsilon^{-1} X_{t_\varepsilon t}\big).$$

Therefore, $(x_\varepsilon^{-1} X_{t_\varepsilon t}^{x,\varepsilon})_{t\ge 0}$ has the same law as $(X_t^{x_\varepsilon^{-1}x,1})_{t\ge 0}$. ■

3 Proof of Theorem 3

Let τ_{γ,x^+} be the random time, defined on the canonical space $C([0,+\infty);\mathbb{R})$:

$$\tau_{\gamma,x^+}(\xi) = \inf\left\{t > 0 : \xi(t) < (1-\gamma)\, x_t^+\right\} \tag{6}$$

(equal to $+\infty$ if this event never happens).

For an initial condition $x \geq x_\varepsilon$, we have $X_0^{x,\varepsilon} > (1-\gamma)\, x_0^+ = 0$ at time zero. Since the processes are continuous, $P_x^\varepsilon(\tau_{\gamma,x^+} > 0) = 1$. For $s \in [0, \tau_{\gamma,x^+}(X^{x,\varepsilon})]$ (all $s \geq 0$ if $\tau_{\gamma,x^+}(X^{x,\varepsilon}) = +\infty$), we have $X_s^{x,\varepsilon} \geq (1-\gamma)\, x_s^+$, so that (by definition of b)

$$b(X_s^{x,\varepsilon}) \geq (1-\gamma)^\alpha\, b\left(x_s^+\right).$$

Therefore, for every $t \in [0, \tau_{\gamma,x^+}(X^{x,\varepsilon})]$,

$$\begin{aligned} X_t^{x,\varepsilon} &= x + \int_0^t b(X_s^{x,\varepsilon})\, ds + \varepsilon W_t \\ &\geq x + (1-\gamma)^\alpha \int_0^t b\left(x_s^+\right) ds + \varepsilon W_t = x + (1-\gamma)^\alpha\, x_t^+ + \varepsilon W_t, \end{aligned}$$

because $x_t^+ = \int_0^t b(x_s^+)\, ds$.

Now consider $\eta \in (0,1)$ such that $1-\eta$ is the mid point between $(1-\gamma)$ and $(1-\gamma)^\alpha$. We have (with equal distance)

$$(1-\gamma) < 1-\eta < (1-\gamma)^\alpha.$$

We rewrite the inequality above in the form

$$\begin{aligned} &X_t^{x,\varepsilon} \geq (1-\eta)\, x_t^+ + R_t^{x,\varepsilon,\gamma}, \\ &R_t^{x,\varepsilon,\gamma} := x + (\gamma-\eta)\, x_t^+ + \varepsilon W_t, \end{aligned}$$

which, we recall, holds for every $t \in [0, \tau_{\gamma,x^+}(X^{x,\varepsilon})]$. Letting $A(x,\varepsilon,\gamma)$ be the event

$$A(x,\varepsilon,\gamma) = \left\{R_t^{x,\varepsilon,\gamma} \geq 0, \quad \forall t \geq 0\right\},$$

we deduce from next lemma that

$$\begin{aligned} &\inf_{x \geq x_\varepsilon} P(A(x,\varepsilon,\gamma)) \geq \lambda_\gamma > 0 \\ &\lim_{\varepsilon \to 0} \inf_{x \geq g(\varepsilon) x_\varepsilon} P(A(x,\varepsilon,\gamma)) = 1, \end{aligned}$$

whenever $g(\varepsilon) \to +\infty$ as $\varepsilon \to 0$. This implies the two claims of the theorem. Indeed, on the event $A(x, \varepsilon, \gamma)$, it holds

$$X_t^{x,\varepsilon} \geq (1-\eta)\, x_t^+ > (1-\gamma)\, x_t^+$$

for every $t \in [0, \tau_{\gamma,x^+}(X^{x,\varepsilon})]$. But this is compatible only with the statement $\tau_{\gamma,x^+}(X^{x,\varepsilon}) = +\infty$. Hence $A(x, \varepsilon, \gamma) \subset \{\tau_{\gamma,x^+}(X^{x,\varepsilon}) = +\infty\}$. The proof of Theorem 3 is complete.

We now prove

Lemma 7 *Given $A > 0$, there is a constant λ_A, independent of ε, such that*

$$P\left(\varepsilon^{\frac{2}{1+\alpha}} + At^{\frac{1}{1-\alpha}} + \varepsilon W_t > 0, \text{ for all } t \geq 0\right) \geq \lambda_A > 0.$$

Moreover, given $g(\varepsilon)$ such that $\lim_{\varepsilon\to 0} g(\varepsilon) = +\infty$,

$$\lim_{\varepsilon\to 0} P\left(\varepsilon^{\frac{2}{1+\alpha}} g(\varepsilon) + At^{\frac{1}{1-\alpha}} + \varepsilon W_t > 0, \text{ for all } t \geq 0\right) = 1.$$

Proof. Since the process $(W_t)_{t\geq 0}$ has the same law as $(\beta^{-1/2} W_{\beta t})_{t\geq 0}$ for every $\beta > 0$, we have

$$\begin{aligned} &P\left(\varepsilon^{\frac{2}{1+\alpha}} g(\varepsilon) + At^{\frac{1}{1-\alpha}} + \varepsilon W_t > 0, \text{ for all } t \geq 0\right) \\ &= P\left(\beta^{\frac{1}{2}} \varepsilon^{\frac{2}{1+\alpha}-1} g(\varepsilon) + \beta^{\frac{1}{2}} At^{\frac{1}{1-\alpha}} \varepsilon^{-1} + W_{\beta t} > 0, \text{ for all } t \geq 0\right). \end{aligned}$$

Choose $\beta = \beta_\varepsilon$ so that $\beta_\varepsilon^{\frac{1}{2}} \varepsilon^{\frac{2}{1+\alpha}-1} = 1$, namely $\beta_\varepsilon^{\frac{1}{2}} = \varepsilon^{-\frac{1-\alpha}{1+\alpha}}$. Then, since $\varepsilon = \beta_\varepsilon^{-\frac{1}{2}\frac{1+\alpha}{1-\alpha}}$,

$$\beta_\varepsilon^{\frac{1}{2}} t^{\frac{1}{1-\alpha}} \varepsilon^{-1} = \beta_\varepsilon^{\frac{1}{2}\frac{1+\alpha}{1-\alpha}} \beta_\varepsilon^{\frac{1}{2}} t^{\frac{1}{1-\alpha}} = \beta_\varepsilon^{\frac{1}{1-\alpha}} t^{\frac{1}{1-\alpha}} = (\beta_\varepsilon t)^{\frac{1}{1-\alpha}}.$$

Therefore

$$\begin{aligned} &P\left(\beta_\varepsilon^{\frac{1}{2}} \varepsilon^{\frac{2}{1+\alpha}-1} g(\varepsilon) + \beta_\varepsilon^{\frac{1}{2}} At^{\frac{1}{1-\alpha}} \varepsilon^{-1} + W_{\beta_\varepsilon t} > 0, \text{ for all } t \geq 0\right) \\ &= P\left(g(\varepsilon) + A(\beta_\varepsilon t)^{\frac{1}{1-\alpha}} + W_{\beta_\varepsilon t} > 0, \text{ for all } t \geq 0\right) \\ &= P\left(g(\varepsilon) + As^{\frac{1}{1-\alpha}} + W_s > 0, \text{ for all } s \geq 0\right). \end{aligned}$$

Whenever $g(\varepsilon) = 1$, the latter probability is positive and independent of ε; we call it λ_A and the first claim of the lemma is proved. Whenever $\lim_{\varepsilon\to 0} g(\varepsilon) = +\infty$, the latter probability tends to one. The proof is complete. ■

4 Proof of Theorem 1

4.1 Selection of the extremal paths

Throughout the proof, we are given a number $\gamma \in (0,1)$. As an easy consequence of Theorem 3 we have:

Corollary 8 *Let $\lambda_\gamma > 0$ be the constant, independent of $\varepsilon \in (0,1)$, given by Theorem 3. Given an infinitesimal function $\widetilde{x}_\varepsilon > x_\varepsilon$ such that $\lim_{\varepsilon\to 0} \widetilde{x}_\varepsilon / x_\varepsilon = +\infty$, there exists an infinitesimal function $\widetilde{t}_\varepsilon > 0$ such that*

$$\inf_{|x|\geq x_\varepsilon} P_x^\varepsilon\left(\tau_{\widetilde{x}_\varepsilon} \leq \widetilde{t}_\varepsilon\right) \geq \lambda_\gamma > 0.$$

Moreover, there exists an infinitesimal function $\widetilde{t}'_\varepsilon > 0$ such that

$$\lim_{\varepsilon\to 0} P_{\pm x_\varepsilon}^\varepsilon\left(\tau_{\widetilde{x}_\varepsilon} \leq \widetilde{t}'_\varepsilon\right) = 1.$$

Here we call an infinitesimal function a function of ε which tends to 0 with ε.

Proof. We know from Theorem 3 that

$$\inf_{x\geq x_\varepsilon} P\left(X_t^{x,\varepsilon} \geq (1-\gamma)\, x_t^+, \quad \forall t \geq 0\right) \geq \lambda_\gamma > 0.$$

The function $(1-\gamma)\, x_t^+$ is equal to $\widetilde{x}_\varepsilon$ at some time $\widetilde{t}_\varepsilon$, and $\widetilde{t}_\varepsilon$ is infinitesimal if $\widetilde{x}_\varepsilon$ is so, that is $\widetilde{t}_\varepsilon \to 0$ if $\widetilde{x}_\varepsilon \to 0$ with ε. Then, by continuity of the trajectories of $(X_t^{x,\varepsilon})_{t\geq 0}$,

$$P_x^\varepsilon\left(\tau_{\widetilde{x}_\varepsilon} \leq \widetilde{t}_\varepsilon\right) \geq P\left(X_t^{x,\varepsilon} \geq (1-\gamma)\, x_t^+, \quad \forall t \geq 0\right) \geq \lambda_\gamma,$$

for $x \geq x_\varepsilon$. Using in a similar argument for initial conditions in $(-\infty, -x_\varepsilon]$, the first assertion is proved.

We now prove the second assertion. We mimic the proof of Proposition 2. For any integer $n \geq 1$, we compute

$$\begin{aligned}
&\max_{\eta=\pm} P_{\eta x_\varepsilon}^\varepsilon\left(\tau_{\widetilde{x}_\varepsilon} \geq 2n\widetilde{t}_\varepsilon\right) \\
&\leq \max_{\eta=\pm} P_{\eta x_\varepsilon}^\varepsilon\left(\tau_{\widetilde{x}_\varepsilon} \geq 2(n-1)\widetilde{t}_\varepsilon\right) \sup_{|x|\geq x_\varepsilon} P_x^\varepsilon\left(\tau_{\widetilde{x}_\varepsilon} \geq 2\widetilde{t}_\varepsilon\right) \\
&\quad + \max_{\eta=\pm} P_{\eta x_\varepsilon}^\varepsilon\left(\tau_{\widetilde{x}_\varepsilon} \geq 2(n-1)\widetilde{t}_\varepsilon\right) \sup_{|x|\leq x_\varepsilon} P_x^\varepsilon\left(\tau_{\widetilde{x}_\varepsilon} \geq 2\widetilde{t}_\varepsilon\right) \\
&:= T_1 + T_2.
\end{aligned}$$

By the first assertion of the statement, the first term in the right-hand side is less than

$$T_1 \le (1-\lambda_\gamma) \max_{\eta=\pm} P^\varepsilon_{\eta x_\varepsilon}\big(\tau_{\widetilde{x}_\varepsilon} \ge 2(n-1)\widetilde{t}_\varepsilon\big).$$

Now, by Proposition 2, the second term is less than

$$\begin{aligned} T_2 &\le \max_{\eta=\pm} P^\varepsilon_{\eta x_\varepsilon}\big(\tau_{\widetilde{x}_\varepsilon} \ge 2(n-1)\widetilde{t}_\varepsilon\big) \sup_{|x|\le x_\varepsilon} P^\varepsilon_x\big(\tau_{x_\varepsilon} > \widetilde{t}_\varepsilon\big) \max_{\eta=\pm 1} P^\varepsilon_{\eta x_\varepsilon}\big(\tau_{\widetilde{x}_\varepsilon} \ge \widetilde{t}_\varepsilon\big) \\ &\le (1-\lambda_\gamma)^2, \end{aligned}$$

for ε small enough. This proves that there exists a constant $c>0$, independent of ε, such that for ε small enough

$$\max_{\eta=\pm} P^\varepsilon_{\eta x_\varepsilon}\big(\tau_{\widetilde{x}_\varepsilon} \ge 2n\widetilde{t}_\varepsilon\big) \le c^n.$$

Choosing $n = \widetilde{t}_\varepsilon^{\,-1/2}$, we complete the proof. ■

Now we can prove that the limit measures of $\{P^\varepsilon_0\}_{\varepsilon\in(0,1)}$ only charge x^+ and x^-. With the same notations as above we define the event

$$\begin{aligned} B^\varepsilon &= \Big\{X^{0,\varepsilon}_{2\widetilde{t}_\varepsilon+t} \ge (1-\gamma)\, x^+_t \text{ for all } t\ge 0\Big\} \\ &\quad \cup \Big\{X^{0,\varepsilon}_{2\widetilde{t}_\varepsilon+t} \le (1-\gamma)\, x^-_t \text{ for all } t\ge 0\Big\} \\ &= \Big\{\max\big(\tau_{\gamma,x^+}(X^{0,\varepsilon}_{2\widetilde{t}_\varepsilon+\cdot}), \tau_{\gamma,x^+}(X^{0,\varepsilon}_{2\widetilde{t}_\varepsilon+\cdot})\big) = +\infty\Big\}, \end{aligned}$$

with the same definition as in (6).

By strong Markov property and Proposition 2, there exists an infinitesimal function δ_ε such that

$$P\big(B^\varepsilon\big) \ge (1-\delta_\varepsilon) \inf_{\eta=\pm} P\Big(\max\big(\tau_{\gamma,x^+}(X^{\eta x_\varepsilon,\varepsilon}_{\widetilde{t}_\varepsilon+\cdot}), \tau_{\gamma,x^-}(X^{\eta x_\varepsilon,\varepsilon}_{\widetilde{t}_\varepsilon+\cdot})\big) = +\infty\Big).$$

Using the second assertion in Corollary 8 and modifying δ_ε, we deduce that

$$P\big(B^\varepsilon\big) \ge (1-\delta_\varepsilon) \inf_{\eta=\pm} P\big(\tau_{\gamma,x^\eta}(X^{\eta\widetilde{x}_\varepsilon,\varepsilon}) = +\infty\big).$$

By the second claim of Theorem 3, we deduce that that

$$\lim_{\varepsilon\to 0} P\left(B^\varepsilon\right) = 1.$$

This implies that any weak limit μ of P^ε_0 is concentrated on the extremal solutions $\pm x^+_\cdot$.

4.2 Weights of the extremal paths

The computation of the weights relies on the following martingale property

Lemma 9 *Define the functions*

$$v(x) = \exp\left(-\frac{2}{1+\alpha}|x|^{1+\alpha}\right),$$
$$V(x) = \int_0^x v(r)dr,$$

for $x \in \mathbb{R}$, together with

$$U_\varepsilon(x) = \begin{cases} \left(\frac{A^+}{\varepsilon^2}\right)^{-\frac{1}{1+\alpha}} V\left(\left(\frac{A^+}{\varepsilon^2}\right)^{\frac{1}{1+\alpha}}\right), & x \geq 0, \\ \left(\frac{A^-}{\varepsilon^2}\right)^{-\frac{1}{1+\alpha}} V\left(\left(\frac{A^-}{\varepsilon^2}\right)^{\frac{1}{1+\alpha}}\right), & x < 0, \end{cases}$$

Then, the process

$$\left(M_t^\varepsilon = U_\varepsilon\left(X_t^{0,\varepsilon}\right)\right)_{t\geq 0}$$

is a martingale.

Before we prove Lemma 9, we first explain how it applies to the end of the proof of Theorem 1.

As in the statement of Theorem 3, we consider an infinitesimal function $\widetilde{x}_\varepsilon$ such that $\widetilde{x}_\varepsilon / x_\varepsilon \to +\infty$ as $\varepsilon \to 0$. We know that $\tau_{\widetilde{x}_\varepsilon}(X^{0,\varepsilon})$ is finite and that $(M_t^\varepsilon)_{t\geq 0}$ is a bounded martingale (V is obviously bounded). By Doob's theorem, we deduce that

$$E\left[M^\varepsilon_{\tau_{\widetilde{x}_\varepsilon}(X^{0,\varepsilon})}\right] = 0,$$

so that

$$U_\varepsilon(\widetilde{x}_\varepsilon) P\left(X^{0,\varepsilon}_{\tau_{\widetilde{x}_\varepsilon}} = \widetilde{x}_\varepsilon\right) + U_\varepsilon(-\widetilde{x}_\varepsilon) P\left(X^{0,\varepsilon}_{\tau_{\widetilde{x}_\varepsilon}} = -\widetilde{x}_\varepsilon\right) = 0,$$

that is

$$\left(A^+\right)^{-\frac{1}{1+\alpha}} V\left(\left(A^+\right)^{\frac{1}{1+\alpha}} \frac{\widetilde{x}_\varepsilon}{x_\varepsilon}\right) P\left(X^{0,\varepsilon}_{\tau_{\widetilde{x}_\varepsilon}} = \widetilde{x}_\varepsilon\right)$$
$$+ \left(A^-\right)^{-\frac{1}{1+\alpha}} V\left(-\left(A^-\right)^{\frac{1}{1+\alpha}} \frac{\widetilde{x}_\varepsilon}{x_\varepsilon}\right) P\left(X^{0,\varepsilon}_{\tau_{\widetilde{x}_\varepsilon}} = -\widetilde{x}_\varepsilon\right) = 0.$$

By Theorem 3, we can let ε tend to zero and then deduce that any limit measure μ must satisfy

$$\big(A^+\big)^{-\frac{1}{1+\alpha}} V(+\infty)\mu(x_\cdot^+) + \big(A^-\big)^{-\frac{1}{1+\alpha}} V(-\infty)\mu(x_\cdot^-) = 0,$$

which completes the proof since $V(+\infty) = V(-\infty)$.

It now remains to prove Lemma 9:

Proof. We notice that the function U_ε is continuously differentiable with

$$U_\varepsilon'(x) = \begin{cases} v\bigg(\big(\dfrac{A^+}{\varepsilon^2}\big)^{\frac{1}{1+\alpha}} x\bigg), & x \geq 0, \\ v\bigg(\big(\dfrac{A^-}{\varepsilon^2}\big)^{\frac{1}{1+\alpha}} x\bigg), & x < 0. \end{cases}$$

The function v is also continuously differentiable with

$$v'(x) = \begin{cases} 2|x|^\alpha v(x), & x \geq 0, \\ -2|x|^\alpha v(x), & x < 0. \end{cases}$$

Therefore, the function U_ε is continuously differentiable on $(-\infty, 0)$ and $(0, +\infty)$ and

$$U_\varepsilon'(x) = \begin{cases} 2\dfrac{A^+}{\varepsilon^2}|x|^\alpha U_\varepsilon(x), & x > 0, \\ -2\dfrac{A^-}{\varepsilon^2}|x|^\alpha U_\varepsilon(x), & x < 0, \end{cases}$$

which proves that, for any $x \neq 0$,

$$\frac{\varepsilon^2}{2} U_\varepsilon''(x) + b(x) U_\varepsilon'(x) = 0.$$

The martingale property follows from Itô's formula. ■

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